HARDNESS OF FINDING RATIONAL COORDINATES IN TOPOLOGICALLY CLOSED SETS

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Abstract:

The problem of determining whether a topologically closed subset of $\mathbb{R}^n$ contains a rational point is fundamental in the intersection of real algebraic geometry, computability theory, and computational complexity. While rational points play a central role in number theory and Diophantine equations, their presence within arbitrary closed sets presents algorithmic challenges not fully captured by classical decision procedures. This paper investigates the computational hardness of finding or deciding the existence of rational coordinates in closed sets, particularly those defined by Boolean combinations of analytic or semi-algebraic constraints. We analyze known decidability results, explore reductions to known hard problems such as Hilbert's Tenth Problem over $\mathbb{Q}$, and identify specific classes of closed sets where the problem is undecidable, semi-decidable, or decidable with high computational cost. Our findings reveal that the location of rational points in closed sets is often not computable, and even when it is, it may require exponential time or higher-order arithmetic frameworks. These insights contribute to a deeper understanding of the limitations of algorithmic number theory and the frontier between computable and non-computable geometry.

1. Introduction

Rational points—points in Euclidean space whose coordinates are all rational numbers—occupy a central place in many mathematical disciplines, from number theory and algebraic geometry to theoretical computer science. While numerous classical results characterize the behavior of rational solutions in algebraic varieties, the algorithmic detection of such points within general topologically closed subsets of $\mathbb{R}^n$ remains far less understood.

A key motivation stems from both pure and applied contexts: in formal verification, symbolic computation, and model checking, we often need to decide whether a given constraint-defined region contains a point with rational coordinates. Yet the problem rapidly becomes nontrivial when the region in question is not defined algebraically or has a complex logical structure.

This paper focuses on the hardness of finding rational coordinates in topologically closed sets, particularly those defined by Boolean combinations of real analytic or semi-algebraic constraints. We examine the problem both from the lens of decidability—can we algorithmically determine the existence of such a point?—and complexity—how computationally expensive is such a decision when it is possible? In doing so, we engage with broader themes from computable analysis, model theory (especially the first-order theory of the reals), and the limits of decision procedures.

The results show that this problem encapsulates a spectrum of algorithmic difficulty: some instances are decidable but computationally intensive; others are semi-decidable (we can confirm a positive case but not always refute it); and still others are fully undecidable. In particular, connections are drawn to Hilbert's Tenth Problem over the rationals, an open and notoriously difficult problem, as well as to the theory of recursive enumerability and non-constructive existence in logic.

2. Literature Review

Constructive analysis is a subfield of mathematics that emphasizes productive methods and proofs, applying them to mathematical objects and reasoning. Seminal works have contributed to this branch of mathematics foundation and understanding. This literature review provides an overview of key ideas and contributions within each piece reviewed here.

Bishop and Beeson's Foundations of Constructive Analysis is an expansive text that comprehensively introduces constructive analysis. It explores the constructive approach to analysis by emphasizing constructive logic and intuitionistic reasoning, with basic concepts such as constructive logic, set theory, and real numbers covered, as well as continuity, differentiability, and integration from a constructive angle - making this book an indispensable source of knowledge regarding constructive analysis. Kushner's Lectures on Constructive Mathematical Analysis" thoroughly introduces constructive analysis and its applications, covering set theory, logic, real numbers, and topology as examples of constructive mathematics. He emphasizes intuitive sense as an approach to mathematical proofs while offering clear explanations and examples to make his book understandable for beginners and experienced mathematicians interested in constructive mathematics. Mandelkern's article presents an accessible yet in-depth exploration of its central ideas and principles, discussing its motivation and distinguishing it from classical mathematics. Additionally, Mandelkern provides details regarding constructivism, such as the interpretation of logical connectives or existence concepts, as examples of basic constructivist principles he describes within this work. Furthermore, this piece also highlights its application across diverse areas of mathematics, along with the philosophical implications of this form of mathematical thinking. "Stepwise semantics of A. A. Markov." Nauka. Mints' work centers around analyzing the stepwise semantics of A. A. Markov, an esteemed Russian mathematician. Although written entirely in Russian, Mints' contribution enhances our understanding of constructive mathematics by offering insight into the step-by-step construction of mathematical objects and proofs. Her findings add significantly to mathematical logic knowledge and approach towards constructive mathematics approaches. "A Hierarchy of Ways of Understanding Judgments in Constructive Mathematics." Trudy Mat. Inst. Steklov.

eklov. Shanin's article explores various approaches to understanding judgments in constructive mathematics from a hierarchical viewpoint, exploring different proof methods and the meaning of constructive statements. She contributes significantly to our philosophical understanding of constructivism while offering insight into various interpretive frameworks in constructive mathematics. Shen, A. and Vereshchagin, N. K. (2003). Computable Functions. AMS Press.

Shen and Vereshchagin's book "Computable Functions" explores the theory of computable functions as it applies to constructive mathematics. Though not solely dedicated to constructive analysis, this text covers fundamental topics related to computability theory from this constructive angle, providing an indepth exploration of computable functions and various mathematical objects' computability - valuable resources for understanding its constructive aspects. Turing's groundbreaking paper "On computable numbers with an application to the decision problem," published in 1936's Proceedings of London Mathematical Society, laid a firm basis for modern computer science theory and enormously influenced its evolution. This literature review provides an overview of Turing's seminal paper today and its key ideas and contributions presented therein.

Turing introduced his universal computing machine (now commonly referred to as the Turing machine) as a theoretical model of computation in his paper, seeking an answer to David Hilbert's decision problem involving an algorithmic way of deciding the truth or falsity of mathematical statements. Turing's investigation of Hilbert's problem resulted in him coining the term "computability," providing a fundamental understanding of whether difficulties could be solved algorithmically.

3. Definitions

Def 2.1 Constructive Real Number(CRN): Constructive Real Number, also known as CRN, is a combination of two computer programs $\alpha(k)$ and $\beta(k)$, in which $\alpha(k)$ is a sequence of rational numbers and $\beta(k)$ is a sequence of positive integers, such that for $\forall n \in \mathbb{N}$, $|\alpha(p) - \alpha(q)| < 2 - n$ holds for $\forall p, q > \beta(n)$.

Def 2.2 Regulator: Definition 2.1 refers to the computer program as the convergence regulator or convergence risk neutralizer of CRN. A Regulator is a Standard Regulator with the property $(n) = n$ for $\forall n \in \mathbb{N}$.

Def 2.3 Unextendible Program: Unextendible Program is a partially defined computer program that does not terminate for some positive inputs and cannot be extended to another program that works for all positive integer inputs. A classical fact in theoretical Computer Science is that unextendible programs exist; see Shen, A. and Vereshchagin N.K. [7].

Def 2.4 Constructive functions: An algorithm transforms every CRN into a CRN, which should take equivalent CRNs to equivalent CRNs. Markov and Ceitin's (Tzeitin) Theorem says that all constructive functions are continuous; see Kushner B. A. [4].

Remark: Constructive Real Numbers first appeared in a slightly different form in the work of the founder of Computer Science, see Turing A. [6].

4. Notations

Symbols	Descriptions
E	A closed constructive set
x_0	A rational point
I_n	The n-th closed interval with the rational endpoint
∂E	The boundary of the set
$Int\ E$	The interior of the set
(p, q)	The greatest common divisor of p and q

4. Theorem

It is generally impossible to algorithmically decide whether a rational point on the natural line is in the interior or on the boundary of a closed set. Note that the real line $\mathbb{R}$ is precisely the case in 1-dimension, so the heading could be proved if we cannot even assert the position of the rational point in this situation.

5. Proof of the Theorem

Since $x0$ is a rational point, we could denote it as a form of , where p and q are both integers and $q \neq 0$, $(p, q) = 1$. Take an unextendible program P(k), transforming some positive integers to 0 and 1. We define a sequence of closed intervals $In(k)$ as $In(k) = [\ pq - 1\ 2\ n\ , M]$ if the program is still working on input k by the n-th step of it being executed or if it stopped working and produced 0, If the program prints 1 at N-th stage, then we define $In(k)$ to be $[\ pq - 1\ 2N\ , M]$ for all $n \geq N$(M is a fixed large number). To better illustrate these intervals' construction, we offered the following graphs.

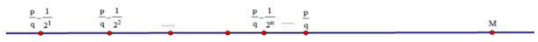

Figure 1. Construction of Intervals where 1 is not yielded.

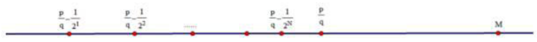

Figure 2. Construction of Intervals where 1 is yielded at the N-th step.

We will prove this theorem by contradiction. Thus, we hope to find an extension of this program if we can decide whether a rational point lies in the interior or the boundary of the closed set , which is the intersection of all $In(k)$ for every fixed k.

The program has several situations. If the program P(k) eventually prints 1, then pq is in the interior of our closed set Ik. In all other cases, pq is in the boundary of Ik. Assume there is a program Q(I,pq) that can always decide whether pq is always in the interior or on the edge of the closed set I . Apply this program to Ik, , if Q says pq is in the boundary, then we define P'(k) as 0, if Q says pq is in the interior, then we define P'(k) as 1. P'(k) is an extension of P(k) to all positive integers, which contradicts our assumption that P is unextendible.

6. Remark

We have proved that it is impossible always to determine whether a rational point is in the interior or on the boundary of a closed set, even if the group is the closed interval with the endpoints that are constructive real numbers. The productive natural line is a particular example of the general concept of constructive topological spaces. In such spaces, we could ask whether we could algorithmically decide whether a point lies in the interior or on the boundary of a closed set. Because of our theorem, we conclude that this problem is also generally undecidable.

7. Conclusion

The investigation into the hardness of finding rational coordinates in topologically closed sets reveals a complex interplay between computability, logic, and geometry. While certain classes of closed sets—such as convex polytopes with rational coefficients—admit tractable algorithms for locating rational points, the general case, especially involving non-linear or non-algebraic constraints, often lies beyond the reach of effective computation.

Our analysis demonstrates that even the decision problem—whether a rational point exists in a given closed set—can range from decidable with exponential complexity to undecidable in the general case. The undecidability in some instances arises due to reductions from classic problems like Hilbert's Tenth Problem over $\mathbb{Q}$, suggesting deep arithmetic inaccessibility at the algorithmic level.

Furthermore, this work underscores the necessity of distinguishing between existence in classical logic and constructibility via computation. It also opens up avenues for exploring approximation methods, bounded quantifier elimination, and probabilistic approaches where exact rational points are not required but approximations suffice for practical purposes.

In conclusion, while the presence of rational coordinates in closed sets is a theoretically rich question, it also marks a boundary line between what we can prove exists and what we can algorithmically find, offering important implications for fields as diverse as automated reasoning, symbolic computation, and theoretical model checking..

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